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Shear layers in a rotating stratified fluid with bottom topography

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The structure of shear layers due to bottom topography in a rotating stratified fluid is obtained under the restriction $\sigma S \ll E^{\frac{1}{2}}$, where $\sigma S = \nu\alpha g\Delta T/K\Omega^2 L$ is a measure of the stratification and $E = \nu/\Omega^2 L$ is the Ekman number. The layers are found to be similar to the side-wall layers discussed by Barcilon & Pedlosky (1967*b*) if $\sigma S \gg E^{\frac{2}{3}}$ and are Stewartson layers if $\sigma S \ll E^{\frac{2}{3}}$. Some comments are made on the possibility of Taylor column formation in a stratified fluid.

1. Introduction

We consider here the shear layers introduced by a discontinuity of bottom slope in a rotating stratified fluid. Our results are of interest for the study of ocean motion over a non-level bottom (e.g. near a ridge or a shelf); however, they are not directly applicable to this in consequence of our implicit assumption that the vertical and horizontal scales have the same order of magnitude (see below). The layers can also appear in laboratory experiments.

Shear layers of the type considered here are to be expected wherever there is a discontinuity on a boundary surface (that is not parallel to the axis of rotation) such that the Ekman-layer transports and the interior fields are discontinuous. They were studied originally by Stewartson (1957), who considered a homogeneous rotating fluid and showed that large gradients may exist in two layers parallel to the axis of rotation.† The thicknesses of these layers are of order $E^{\frac{1}{4}}L$ and $E^{\frac{1}{3}}L$, where $E = \nu(\Omega L^2)^{-1}$ is the Ekman number and ν, Ω and L are the kinematic viscosity, the angular velocity and a characteristic length, respectively. These layers may be either boundary layers on a wall or detached shear layers; outside them the flow is geostrophic and controlled by the Ekman layers (of thickness $E^{\frac{1}{2}}L$).

Barcilon & Pedlosky (1967*b*, hereafter referred to as BP) have shown that the side-wall layers are progressively altered as the stratification increases, but they did not discuss the detached shear layers. A measure of the stratification is $\sigma S = \sigma N^2\Omega^{-2}$, where σ is the Prandtl number and N the Brunt–Väisälä frequency. When $\sigma S \ll E^{\frac{2}{3}}$, the fluid behaves essentially as if it were homogeneous; if $E^{\frac{2}{3}} \ll \sigma S \ll E^{\frac{1}{2}}$ the $E^{\frac{1}{3}}$ layer splits into a hydrostatic layer of thickness $\sigma S^{\frac{1}{2}}$ and a buoyancy layer of thickness $E^{\frac{1}{2}}\sigma S^{-\frac{1}{4}}$; however, the latter is not present to lowest

† It should be kept in mind that *physically* there is just *one* layer, one transition region. *Mathematically* we can distinguish, asymptotically for small values of the relevant parameters, different regions where appropriate approximations to the equations are taken, corresponding to particular dynamical balances.

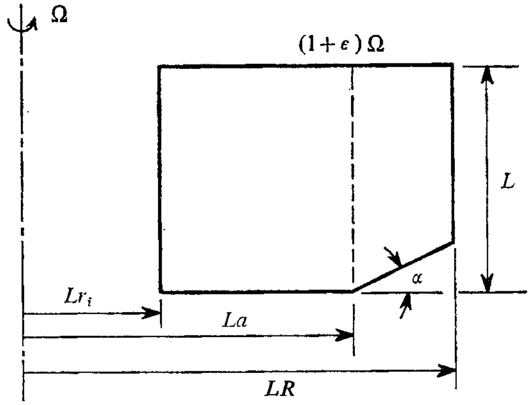

FIGURE 1

order in the particular case (no heat flux through the wall) treated by BP. The parametric range $E^{\frac{1}{2}} \ll \sigma S \ll 1$ has not been investigated in detail, although Barcilon & Pedlosky suggest that two layers of thicknesses $E^{\frac{1}{4}}\sigma S^{-\frac{1}{4}}$ and $(\sigma S)^{\frac{1}{2}}$ exist.

The structure and the role of the wall layers of BP depend on the boundary conditions; if a vertically varying heat flux is imposed on the vertical wall, the influence is felt in the interior, which is no longer controlled by the Ekman layers, and the buoyancy layer will play a significant role. We show here that, although there is a heat flux through the free shear layers, there is still no buoyancy layer and no influence in the interior.

In §2 we follow the formulation of BP and, under the assumptions σS, $E \ll 1$, we calculate the interior fields without referring to the side-wall or the shear layers; this is possible in virtue of the result of BP that in this parametric range the motion is controlled by the Ekman layers. The structure of the shear layers is derived in §3, under the restriction $\sigma S \ll E^{\frac{1}{2}}$, and the solution is completed in §4, using the method of matched asymptotic expansions. For $\sigma S \gg E^{\frac{2}{3}}$ it is found that the results are quite similar to the BP solution for the wall layers; for $\sigma S \ll E^{\frac{2}{3}}$ the layers are essentially Stewartson layers and are given in §5. In both cases the layer has a sandwich structure, a thin layer between two thicker ones. The mass transport in the thicker ($E^{\frac{1}{4}}$) layers vanishes somewhere between the top and bottom, so that the two types of layers interchange fluid and they both take part in the transport of fluid from top to bottom. The results for the $E^{\frac{1}{4}}$ layer are shown to be uniformly valid for $\sigma S \ll E^{\frac{1}{2}}$.

In the study of an ocean model the effects of different horizontal and vertical scales must be taken into account. Blumsack (1972) has shown that there are situations where vertical layers of thickness $E^{\frac{1}{4}}D$ and $\sigma S^{\frac{1}{2}}D$ (D being the depth) can be predicted in a rotating stratified ocean.

2. Formulation

We consider the flow of an incompressible, viscous, heat conducting fluid in the axially symmetric container depicted in figure 1. The system is rotating with angular velocity Ω, the motion being driven by a differential rotation of the top plate. The height of the container is L. A rotating cylindrical co-ordinate system with unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ is used, the bottom being described by

$$z^* = \begin{cases} 0 & \text{for} \quad r^* < aL, \\ (r^*-aL)\tan\alpha & \text{for} \quad aL < r^* < RL. \end{cases}$$

Starred quantities are dimensional. The velocity components are (u^*, v^*, w^*); the lid $(z^* = L)$ has arbitrary relative velocity $v_T^*(r^*)\,\hat{\boldsymbol{\theta}}$.

For convenient reference we use the formulation, approximations and notation of BP† and start with their non-dimensional equations (2.1)–(2.3):

$$2\hat{\mathbf{k}}\times\mathbf{q} = -\nabla p + T\hat{\mathbf{k}} + E\nabla^2\mathbf{q}, \tag{2.1}$$

$$\nabla.\mathbf{q} = 0, \tag{2.2}$$

$$\sigma S\hat{\mathbf{k}}.\mathbf{q} = E\nabla^2 T. \tag{2.3}$$

Our typical velocity is $|v_T^*(r^*)|_{\max}$, yielding the Rossby number

$$\epsilon = |v_T^*(r^*)|_{\max}/\Omega L.$$

The system (2.1)–(2.3) is to be solved subject to the boundary conditions

$$\left.\begin{aligned} \mathbf{q}(r,0) &= 0, \quad z = 0, \quad r < a, \\ \mathbf{q}(r,(r-a)\tan\alpha) &= 0, \quad z = (r-a)\tan\alpha, \quad a < r < R, \\ \mathbf{q}(r,1) &= v_\tau(r)\,\hat{\boldsymbol{\theta}}, \quad z = 1, \end{aligned}\right\} \tag{2.4}$$

and $\hat{\mathbf{n}}.\nabla T = 0$ on all solid boundaries.

We look for axisymmetric solutions; accordingly, (2.2) is satisfied by letting

$$\mathbf{q} = \nabla\times\left(\frac{1}{r}\psi(r,z)\,\hat{\boldsymbol{\theta}}\right) + \frac{1}{r}\chi(r,z)\,\hat{\boldsymbol{\theta}},$$

whereby (2.1) and (2.3) become

$$2\partial_z\chi = E\mathscr{L}^4\psi + r\,\partial_r T, \tag{2.5}$$

$$-2\,\partial_z\psi = E\mathscr{L}^2\chi, \tag{2.6}$$

$$\sigma S\,\partial_r\psi = E(\partial_r r\,\partial_r T + \partial_z^2 rT), \tag{2.7}$$

where $\mathscr{L}^2 \equiv r\,\partial_r(r^{-1}\partial_r) + \partial_z^2 \equiv \mathscr{R}^2 + \partial_z^2$. The boundary conditions (2.4) become

$$\chi = \begin{cases} 0 & \text{at} \quad z = (r-a)\tan\alpha H(r-a), \\ rv_\tau(r) \equiv \chi_\tau(r) & \text{at} \quad z = 1 \end{cases}$$

and $\psi = \partial_n\psi = \partial_n T = 0$ on all boundaries. (H is Heaviside's step function.)

The z derivatives in the viscous terms of (2.5)–(2.7) are important only in the Ekman layers on the top and on the bottom. The Ekman-layer analysis is

† This implies, among other things, that the temperature is written as

$$T^* = T_0^* + \Delta TZ^*/L + \epsilon(\Omega^2L/\alpha g)T,$$

so that the basic stratification is linear. In the undisturbed state the isopycnal surfaces are then horizontal provided that $Fr = \Omega^2L/g \ll 1$ (Greenspan 1968, §1.4; Barcilon & Pedlosky 1967*c*), and provided that the sloping part of the bottom is maintained at the appropriate temperature.

standard and need not be reproduced; the constraints imposed on the interior flow† are:

at $z = 1$:

$$\psi + \tfrac{1}{2}E^{\frac{1}{2}}(\chi - \chi_T) = 0, \quad \partial_z T = \sigma S E^{-\frac{1}{2}} r^{-1}\partial_r \psi; \tag{2.8 a, b}$$

at $z = 0$, $r < a$:

$$\psi - \tfrac{1}{2}E^{\frac{1}{2}}\chi = 0, \quad \partial_z T = -\sigma S E^{-\frac{1}{2}} r^{-1}\partial_r \psi; \tag{2.9 a, b}$$

at $z = (r-a)\tan\alpha$, $r > a$:

$$\psi - \tfrac{1}{2}(E\cos\alpha)^{-\frac{1}{2}}\chi = 0, \tag{2.10 a}$$

$$\partial_n T = -\sigma S E^{-1} r^{-1}[(\sin\alpha)\psi + (E\cos\alpha)^{\frac{1}{2}}\partial_n \psi], \tag{2.10 b}$$

where $\mathbf{n}$ is the inward normal to the wall. Equations (2.10) reduce to (2.9) for $\alpha = 0$ and are invalid as $\tan\alpha \to \infty$; when $\sin\alpha = O(1)$, (2.10 *b*) is, to leading order,

$$r(\sin\alpha\,\partial_r T - \cos\alpha\,\partial_z T) = \sigma S E^{-1}\sin\alpha\,\psi. \tag{2.10 c}$$

We can now obtain asymptotically valid solutions of (2.5)–(2.7) using boundary-layer methods. The notation is as follows: the subscripts + or − denote the fields in $r > a$ or $r < a$; the regions away from $r = a$ and $r = R$ are called the interior, wherein we add the subscript i (i.e. $\psi_{+i}, \psi_{-i}, \chi_{+i} \ldots$). In the neighbourhood of $r = a$ we consider additive corrections to the interior fields, vanishing exponentially as $|r-a|$ becomes large; for those boundary-layer corrections we use the overbar, caret and tilde of BP.

The interior

It will be shown at the end of §4 that by taking $\partial_z T_{\pm i} = 0$ and $\sigma S\psi_i = Er\,\partial_r T_i$ the equations and boundary conditions on T are satisfied; then (2.5)–(2.7) reduce to

$$2\,\partial_z \chi_i = r\,\partial_r T_i + O(E^{\frac{3}{2}}), \tag{2.11 a}$$

$$\partial_z \psi_i = O(E), \tag{2.11 b}$$

$$\sigma S\,\partial_r \psi_i = E\,\partial_r r\,\partial_r T_i. \tag{2.11 c}$$

The solution is straightforward; using the boundary conditions (2.8) and (2.9) for $r < a$, we obtain

$$\chi_{-i}(r,z) = \tfrac{1}{2}\chi_T(r) + \frac{\sigma S}{8E^{\frac{1}{2}}}\,\frac{\chi_T(r)}{1+\sigma S/8E^{\frac{1}{2}}}\,(z-\tfrac{1}{2}), \tag{2.12}$$

$$\psi_{-i}(r,z) = \frac{E^{\frac{1}{2}}}{4}\,\frac{\chi_T(r)}{1+\sigma S/8E^{\frac{1}{2}}}, \tag{2.13}$$

$$T_{-i}(r,z) = \frac{\sigma S}{4E^{\frac{1}{2}}}\,\frac{1}{1+\sigma S/8E^{\frac{1}{2}}}\int_0^r \chi_T(r')\,\frac{1}{r'}\,dr'. \tag{2.14}$$

These solutions have been given in a footnote in BP; they are valid when there is no heat flux through the boundaries. We see that provided that $\sigma S \ll 1$ the interior flow can be obtained, to $O(\sigma S)$, without calculation of the side-wall layers.

Similarly, using (2.10) instead of (2.9) we have in $r > a$

$$\psi_{+i}(r,z) = \tfrac{1}{2}E^{\frac{1}{2}}\lambda\chi_T(r)/D, \tag{2.15}$$

† For this purpose an 'interior region' is any region where the radial gradients are $o(E^{-\frac{1}{2}})$.

$$\chi_{+i}(r,z) = \chi_\tau(r)\left[\frac{1}{1+\lambda}+\frac{\lambda\sigma S}{4E^{\frac{1}{2}}D}\left(z-\frac{\lambda z_B+1}{1+\lambda}\right)\right], \quad (2.16)$$

$$T_{+i}(r,z) = \frac{\sigma S}{2E^{\frac{1}{2}}}\frac{\lambda}{D}\int_0^r \chi_\tau(r')\frac{1}{r'}\,dr', \quad (2.17)$$

where $\lambda \equiv (\cos\alpha)^{-\frac{1}{2}}, \quad z_B = z_B(r) \equiv (r-a)\tan\alpha$

and $D \equiv 1+\lambda+\lambda(\sigma S/4E^{\frac{1}{2}})(1-z_B).$

The main features are well known (Barcilon & Pedlosky 1967 *a, b*): for $\sigma S \ll E^{\frac{1}{2}}$ the flow is to leading order the same as in the homogeneous case; when $E^{\frac{1}{2}} \ll \sigma S \ll 1$ vertical motions are constrained to become of $O(E/\sigma S)$ and the swirl velocity increases linearly from its value at the bottom to its value at the top; when $\sigma S = O(1)$ there is no Ekman suction and the flow is controlled by dissipative processes.

From (2.12)–(2.17) we see that the fields are discontinuous at $r = a$, except when $\sigma S \gg E^{\frac{1}{2}}$; to smooth out the discontinuities there must be a narrow transition region where the radial gradients are large and the approximate equations (2.11) are invalid.

3. The shear layers

The correction fields near $r = a$ satisfy

$$2\partial_z\chi_c = E\partial_r^4\psi_c + a\partial_r T_c, \quad (3.1a)$$

$$-2\partial_z\psi_c = E\partial_r^2\chi_c, \quad \sigma S\partial_r\psi_c = aE\partial_r^2 T_c. \quad (3.1b, c)$$

Eliminating T_c and ψ_c, we obtain an equation for χ_c:

$$(E^2\partial_r^6+\sigma S\partial_r^2+4\partial_z^2)\chi_c = 0, \quad (3.2)$$

with the boundary conditions, derived from a combination of (2.8)–(2.10) and (3.1),

$$4E^{\frac{1}{2}}\partial_z\chi_{-c} \pm (\sigma S+E^2\partial_r^4)\chi_{-c} = 0 \quad (r < a,\ z = \tfrac{1}{2}\pm\tfrac{1}{2}), \quad (3.3)$$

or

$$4(E\cos\alpha)^{\frac{1}{2}}\partial_z\chi_{+c} - (\sigma S+E^2\partial_r^4)\chi_{+c} = 0 \quad (r > a,\ z \simeq 0), \quad (3.4a)$$

$$4E^{\frac{1}{2}}\partial_z\chi_{+c} + (\sigma S+E^2\partial_r^4)\chi_{+c} = 0 \quad (r > a,\ z = 1). \quad (3.4b)$$

The region $r < a$

$\chi_{-c} = (A_-\sin\omega z+B_-\cos\omega z)\exp\beta(a-r)$ is a solution of (3.2)–(3.3) provided that

$$\omega^2 = \tfrac{1}{4}(E^2\beta^6+\sigma S\beta^2), \quad (3.5)$$

$$\omega^2\left[2\cos\omega+\left(\frac{\omega}{E^{\frac{1}{2}}\beta^2}-\frac{E^{\frac{1}{2}}\beta^2}{\omega}\right)\sin\omega\right] = 0, \quad (3.6)$$

$$A_- = (\omega/E^{\frac{1}{2}}\beta^2)B_-. \quad (3.7)$$

The roots of (3.6) are

$$\omega = 0, \quad \text{i.e.} \quad \beta^4 = -\sigma SE^{-2}, \quad (3.8)$$

and approximately, if $\sigma S \ll E^{\frac{1}{2}}$,

$$(a) \quad \omega = n\pi, \quad \text{i.e.} \quad E^2\beta^6+\sigma S\beta^2 = (2n\pi)^2,$$

which gives

$$\beta^6 = (2n\pi E^{-1})^2 \quad \text{for} \quad \sigma S \ll E^{\frac{2}{3}} \tag{3.9}$$

and

$$\beta^2 = (\sigma S)^{-1}(2n\pi)^2 \quad \text{for} \quad \sigma S \gg E^{\frac{2}{3}}; \tag{3.10}$$

$$(b) \quad \beta^2 = 2E^{-\frac{1}{2}}\left(1-\frac{\epsilon}{24}\right) \quad \text{where} \quad \epsilon \equiv \sigma S E^{-\frac{1}{2}}. \tag{3.11}$$

These different roots correspond to the layers of Stewartson (1957) and BP.

Consider first the modified $E^{\frac{1}{4}}$ layers from (3.11). Noting that $\omega \ll 1$, $\omega^2 = O(\epsilon)$ and $\omega^2/\beta^2 E^{\frac{1}{2}} = O(\epsilon)$, we have

$$\bar{\chi}_- = B_-\{[1-\tfrac{1}{4}\epsilon z(z-1)]\exp[-E^{-\frac{1}{4}}2^{\frac{1}{2}}(1-\tfrac{1}{48}\epsilon)\,|r-a|]+O(\epsilon^2)\}, \tag{3.12}$$

where B_- is a constant and the overbar notation conforms with BP.

Limiting ourselves to the parameter range $E^{\frac{2}{3}} \ll \sigma S \ll E^{\frac{1}{2}}$, we have from the roots (3.10) the hydrostatic layers, where

$$\omega = O(1), \quad \omega^2/\beta^2 E^{\frac{1}{2}} = O(\epsilon);$$

$$\hat{\chi}_- = \sum_{n=1}^{\infty} B_{-n}\{\cos n\pi z \exp[-2n\pi\sigma S^{-\frac{1}{2}}\,|r-a|]+O(\epsilon)\}. \tag{3.13}$$

Finally, we consider the buoyancy layer obtained from the root $\omega = 0$, equation (3.8). This implies that χ is independent of the vertical co-ordinate; such a solution is apparently possible, but then the conditions (3.3) are satisfied in a degenerate way; the buoyancy layer will be discussed in §4.

The region $r > a$

Similarly, $\chi_{+c} = (A_+\sin\mu z + B_+\cos\mu z)\exp\gamma(r-a)$ is a solution of (3.2)–(3.4) provided that

$$\mu^2 = \tfrac{1}{4}(E^2\gamma^6+\sigma S\gamma^2), \tag{3.14}$$

$$\mu^2\left[(1+\lambda)\cos\mu + \left(\frac{\mu\lambda}{\gamma^2 E^{\frac{1}{2}}} - \frac{E^{\frac{1}{2}}\gamma^2}{\mu}\right)\sin\mu\right] = 0, \tag{3.15}$$

$$A_+ = (\mu\lambda/E^{\frac{1}{2}}\gamma^2)\,B_+. \tag{3.16}$$

From this we obtain the $E^{\frac{1}{4}}$ layer, where

$$\gamma^2 = (1+\lambda)E^{-\frac{1}{2}}\left[1-\frac{\epsilon}{12}\left(\frac{\lambda^2-\lambda+1}{\lambda+1}\right)\right], \tag{3.17}$$

$$\bar{\chi}_+ = B_+\left(1-\frac{(1+\lambda)\epsilon}{8}z\left(z-\frac{2\lambda}{1+\lambda}\right)\right)\exp\left[-E^{-\frac{1}{4}}(1+\lambda)^{\frac{1}{2}}\left(1-\frac{\epsilon}{24}\frac{\lambda^2-\lambda+1}{1+\lambda}\right)|r-a|\right], \tag{3.18}$$

and the hydrostatic layer, where

$$\mu = n\pi, \quad \gamma^2 = (\sigma S)^{-1}(2n\pi)^2, \tag{3.19}$$

$$\hat{\chi}_+ = \sum_{n=1}^{\infty} B_{+n}\cos n\pi z\exp[-2n\pi\sigma S^{-\frac{1}{2}}(r-a)]. \tag{3.20}$$

4. Matching of the solutions

To determine the constants $B_\pm$ and $B_{\pm n}$, appropriate jump conditions at $r = a$ must be found. A systematic way to obtain them is by the method of matched asymptotic expansions (Van Dyke 1964). We outline the procedure. The outer solution for $r < a$ is taken as

$$\chi_{0-} \equiv \chi_{-i} + \bar{\chi}_- + \hat{\chi}_-$$

and for $r > a$ as

$$\chi_{0+} \equiv \chi_{+i} + \bar{\chi}_+ + \hat{\chi}_+.$$

The inner solution is obtained by introducing the stretched co-ordinate $\rho = E^{-\frac{1}{2}}\sigma S^{\frac{1}{4}}(r-a)$ in (3.2); the inner field $\tilde{\chi}$ ($\tilde{\chi}$ now represents the total field rather than a boundary-layer correction) is the solution of

$$(\partial_\rho^6 + \partial_\rho^2)\tilde{\chi} = -4E\sigma S^{-\frac{3}{2}}\partial_z^2\tilde{\chi}, \tag{4.1}$$

where $\tilde{\chi}$ is to be expanded in an asymptotic series

$$\tilde{\chi} = \sum_{i=0} \mu_i(E, \sigma S)\tilde{\chi}^{(i)}. \tag{4.2}$$

The asymptotic sequence $\mu_i(E, \sigma S)$ is found from the form of the inner expansion of the outer solution, expressed in inner variables. We have

$$\begin{aligned}\chi_{0-} \sim\ & \tfrac{1}{2}\chi_T(a)\,[1 + \tfrac{1}{4}\epsilon(z - \tfrac{1}{2}) + \ldots] \\ & + B_-[1 - \tfrac{1}{4}\epsilon z(z-1)]\,[1 + \epsilon^{\frac{1}{2}}\delta 2^{\frac{1}{2}}(1 - \tfrac{1}{48}\epsilon)\rho + \epsilon\delta^2\rho^2 + \ldots] \\ & + \sum_{n=1} B_{-n}\cos n\pi z[1 + 2n\pi\delta\rho + \ldots]\end{aligned} \tag{4.3a}$$

and

$$\begin{aligned}\chi_{0+} \sim\ & \frac{\chi_T(a)}{1+\lambda}\left[1 + \frac{\lambda\epsilon}{4}\left(z - \frac{1}{1+\lambda}\right) + \ldots\right] + B_+\left[1 - \tfrac{1}{8}\epsilon(1+\lambda)z\left(z - \frac{2\lambda}{1+\lambda}\right)\right] \\ & \times\left[1 - \epsilon^{\frac{1}{2}}\delta(1+\lambda)^{\frac{1}{2}}\left(1 - \frac{\epsilon}{24}\,\frac{\lambda^2-\lambda+1}{1+\lambda}\right)\rho + \frac{\epsilon\delta^2(1+\lambda)}{2}\rho + \ldots\right] \\ & + \sum_{n=1}^{\infty} B_{+n}\cos n\pi z[(1 - 2n\pi\delta\rho + \ldots],\end{aligned} \tag{4.3b}$$

where $\delta \equiv E^{\frac{1}{2}}\sigma S^{-\frac{3}{4}}$.

The expansion for $\tilde{\chi}$ is then chosen as

$$\tilde{\chi} = \sum_{i,j} (\epsilon^{\frac{1}{2}})^i\,\delta^j\tilde{\chi}_{i,j}. \tag{4.4}$$

Since $\tilde{\chi}$ represents the total field in the neighbourhood of $r = a$, we must require $\tilde{\chi}$, and therefore each $\tilde{\chi}_{i,j}$, to be continuous, with its first five derivatives. Hence, substituting (4.4) into (4.1) and solving yields

$$\tilde{\chi}_{i,j} = \begin{cases} \tilde{A}_{i,j}(z)\,\rho + \tilde{B}_{i,j}(z) \quad \text{for all } i, \quad j = 0, 1, & (4.5) \\ -\tfrac{2}{3}\partial_z^2\tilde{A}_{i,j-2}\rho^3 - 2\,\partial_z^2\tilde{B}_{i,j-2}(z)\rho^2 + \tilde{A}_{i,j}(z)\rho + \tilde{B}_{i,j} \quad \text{for all} \quad i,j = 2,3. & (4.6)\end{cases}$$

In deriving those solutions we have assumed $\partial_z\tilde{\chi} = O(1)$ (in (4.1) only the radial co-ordinate is stretched), so that z appears only parametrically in the equations. Near $z = 0$ and $z = 1$, i.e. where the Ekman layers merge into the shear layers, there is a 'corner region' where the full equations (2.5)–(2.7) must be solved, because the z derivatives can be as large as the r derivatives. Therefore the solutions (4.5) and (4.6) are valid only for $0 < z < 1$ and not at $z = 0$ and $z = 1$. Pedlosky (1971) encounters a similar difficulty in the corresponding problem of a side-wall layer.

Since $(\partial_\rho^6 + \partial_\rho^2)\tilde{\chi}_{i,4} = -4\partial_z^2\tilde{\chi}_{i,2}$, the terms $\tilde{\chi}_{i,4}$ will involve the general solution of $(\partial_\rho^4 + 1)\tilde{\chi} = 0$, and therefore represent a true contribution from the buoyancy layer only if $\partial_z^4\tilde{A}_{i,0}$ or $\partial_z^4\tilde{B}_{i,0}$ are non-zero. It turns out that the first such contribution is from $\tilde{\chi}_{2,4}$.

Clearly (4.5) and (4.6) do not represent boundary-layer contributions but are merely the leading terms of the expansion of $\chi_{0\pm}$ near $r = a$. Matching $\tilde{\chi}$ with χ_{0-} and χ_{0+}, we obtain the constants:

$$B_- = \frac{\chi_r(a)}{2(\lambda+1)}[[2(\lambda+1)]^{\frac{1}{2}} - (\lambda+1)]\left[1 - \frac{\epsilon}{48(\lambda+1)}(2(5\lambda+2) + (2\lambda-1) \times [2(\lambda+1)]^{\frac{1}{2}}) + O(\epsilon^{\frac{3}{2}})\right], \quad (4.7a)$$

$$B_+ = \frac{\chi_r(a)}{2(\lambda+1)}[[2(\lambda+1)]^{\frac{1}{2}} - 2]\left[1 - \frac{\epsilon}{48(\lambda+1)}(2\lambda^2 + 13\lambda - 1 + (2\lambda-1) \times [2(\lambda+1)]^{\frac{1}{2}}) + O(\epsilon^{\frac{3}{2}})\right], \quad (4.7b)$$

$$B_{\pm n} = \frac{\pm\epsilon\chi_r(a)\,(\lambda-1)}{2(\lambda+1)^{\frac{1}{2}}(2n\pi)^2}\left[1 \pm \epsilon^{\frac{1}{2}}\frac{(\lambda+1)^{\frac{1}{2}} - 2^{\frac{1}{2}}}{2n\pi}\right] + O(\epsilon^2). \quad (4.7c)$$

The result (4.5) implies that χ_o and $\partial_r\chi_o$ are continuous at $r = a$; an alternative way to determine the constants $B_\pm$ and $B_{\pm n}$ is by solving

$$\chi_{-i} + \overline{\chi}_- + \hat{\chi}_- = \chi_{+i} + \overline{\chi}_+ + \hat{\chi}_+$$

and

$$\partial_r(\chi_{-i} + \overline{\chi}_- + \hat{\chi}_-) = \partial_r(\chi_{+i} + \overline{\chi}_+ + \hat{\chi}_+)$$

at $r = a$. This is entirely equivalent to the matching procedure, and formally more direct.

The higher derivatives are not, and do not need to be, continuous, to this order. The continuity of the solution can be satisfied by including higher order terms. There will be a buoyancy layer where the azimuthal velocity component is of order $\epsilon\delta^4 = E^{\frac{3}{2}}\sigma S^{-2}$.

The radial and vertical velocity components are continuous and can be derived from the stream function, which is given by

$$\partial_z\psi = -\tfrac{1}{2}E\partial_r^2\chi.$$

We have

$$\overline{\psi}_- = -E^{\frac{1}{2}}B_-(z-\tfrac{1}{2})\left[1 - \frac{\epsilon}{12}z(z-1)\right]\exp\left[E^{-\frac{1}{2}}2^{\frac{1}{2}}\left(1 - \frac{\epsilon}{48}\right)(r-a)\right], \quad (4.8a)$$

$$\overline{\psi}_+ = -\frac{E^{\frac{1}{2}}}{2}(1+\lambda)B_+\left[z - \frac{\lambda}{1+\lambda} - \frac{\epsilon}{24}(1+\lambda)z\left(z^2 - \frac{3\lambda}{1+\lambda}z + 2\frac{\lambda^2-\lambda+1}{(\lambda+1)^2}\right)\right] \times \exp\left[-E^{-\frac{1}{2}}(1+\lambda)^{\frac{1}{2}}\left(1 - \frac{\epsilon}{24}\frac{\lambda^2-\lambda+1}{\lambda+1}\right)(r-a)\right], \quad (4.8b)$$

$$\hat{\psi}_\pm = -\frac{E}{\sigma S}\sum_{n=1}^{\infty} B_{\pm n}(2n\pi)\sin n\pi z \exp\left[\mp 2n\pi\sigma S^{-\frac{1}{2}}(r-a)\right]. \quad (4.8c)$$

Thus both layers take part in the net mass transport from the bottom to the top; this transport is equal to the difference between the flux in the Ekman layers on the sloping and on the flat parts of the bottom. The transport in the thicker outer layers ($E^{\frac{1}{4}}$ layers) vanishes between $z = 0$ and $z = 1$; at those points all the flux is in the inner layer.

Finally, we must show that the temperature field satisfies the boundary and continuity conditions. Since ψ_i is of the form $E^{\frac{1}{2}}(\psi_0+\epsilon\psi_1+\ldots)$, T_i is of the form $\epsilon(T_0+\epsilon T_1+\ldots)$ (see equation (2.7)). Then the conditions (2.8 *b*) and (2.9 *b*) reduce, to leading order, to $\partial_z T_i = 0$ $(z = 0, 1)$, whereas (2.10 *c*) is obviously satisfied by taking $\partial_z T_i = 0$ and $\sigma S\psi_i = Er\,\partial_r T_i$ from (2.11 *c*). Clearly then the solutions (2.14) and (2.17) satisfy the boundary conditions at top and bottom through the Ekman layers. Now one expects the radial heat flux to be continuous at $r = a$ (and zero at $r = R$); since $\partial_r T = r^{-1}\sigma SE^{-1}\psi$ everywhere, the continuity (or, at the wall, the vanishing) of ψ fulfills this requirement.

5. Concluding remarks

The results presented above are valid for $E^{\frac{2}{3}} \ll \sigma S \ll E^{\frac{1}{2}}$. However, the solution for the interior is valid under the milder restriction $\sigma S \ll 1$; when $E^{\frac{1}{2}} \ll \sigma S \ll 1$ the discontinuities in the interior fields appear only to higher orders and the shear layers disappear (see Barcilon & Pedlosky 1967 *a*, *b*). We obtain the solution for a homogeneous fluid by setting $\sigma S = 0$. The buoyancy and hydrostatic layers merge into a $E^{\frac{1}{4}}$ Stewartson layer and the results are (the calculations are not reproduced here; they follow essentially Stewartson (1957), see also Moore & Saffman (1969) and Greenspan (1968))

$$\chi_+(r,z) = \chi_T(r)\,(1+\lambda)^{-1}[1+\tfrac{1}{2}\{[2(\lambda+1)]^{\frac{1}{2}}-2\}\exp\{-(\lambda+1)^{\frac{1}{2}}E^{-\frac{1}{4}}(r-a)\}]+\hat{\chi}(\rho,z), \tag{5.1 a}$$

$$\psi_+(r,z) = \frac{E^{\frac{1}{2}}\chi_T(r)}{2}\left[\frac{\lambda}{1+\lambda}-([2(\lambda+1)]^{\frac{1}{2}}-2)\times\left(z-\frac{\lambda}{1+\lambda}\right)\exp\left(-(\lambda+1)^{\frac{1}{2}}E^{-\frac{1}{4}}(r-a)\right)\right]+\hat{\psi}(\rho,z), \tag{5.1 b}$$

$$\chi_-(r,z) = \tfrac{1}{2}\chi_T(r)\left[1+\frac{1}{\lambda+1}([2(\lambda+1)]^{\frac{1}{2}}-(\lambda+1))\exp\left((\lambda+1)^{\frac{1}{2}}E^{-\frac{1}{4}}(r-a)\right)\right]+\hat{\chi}(\rho,z), \tag{5.2 a}$$

$$\psi_-(r,z) = \frac{E^{\frac{1}{2}}\chi_T(r)}{4}\left[1-\frac{2}{\lambda+1}([2(\lambda+1)]^{\frac{1}{2}}-(\lambda+1))\,(z-\tfrac{1}{2})\exp\left((\lambda+1)^{\frac{1}{2}}E^{-\frac{1}{4}}(r-a)\right)\right]+\hat{\psi}(\rho,z), \tag{5.2 b}$$

where

$$\hat{\chi}(\rho,z) = -\frac{|\rho|}{\rho}E^{\frac{1}{6}}\frac{\chi_T(r)\,(\lambda-1)}{6[2(\lambda+1)]^{\frac{1}{2}}}\sum_{n=1}^{\infty}(2n\pi)^{-\frac{2}{3}}\times[e^{-\gamma_n|\rho|}-2e^{-\frac{1}{2}\gamma_n|\rho|}\cos(\gamma_n\tfrac{1}{2}3^{\frac{1}{2}}|\rho|-\tfrac{1}{3}\pi)]\cos n\pi z, \tag{5.3 a}$$

$$\hat{\psi}(\rho,z) = \frac{|\rho|}{\rho}E^{\frac{1}{2}}\frac{\chi_T(r)\,(\lambda-1)}{3[2(\lambda+1)]^{\frac{1}{2}}}\sum_{n=1}^{\infty}(2n\pi)^{-1}\times[e^{-\gamma_n|\rho|}+2e^{-\frac{1}{2}\gamma_n|\rho|}\cos(\gamma_n\tfrac{1}{2}3^{\frac{1}{2}}|\rho|)]\sin n\pi z, \tag{5.3 b}$$

$$\rho \equiv E^{-\frac{1}{3}}(r-a), \quad \gamma_n \equiv (2n\pi)^{\frac{1}{3}}.$$

We observe that the $E^{\frac{1}{4}}$ layer is obtained simply by letting $\sigma S = 0$ in the corresponding solution for a stratified fluid. We conclude that the results (3.12) and (3.18) (combined with (4.7)) are uniformly valid for $\sigma S \ll E^{\frac{1}{2}}$.

The solution (5.1)–(5.3) is almost identical to Stewartson's result (1957; see also Greenspan 1968, p. 104) for a concentric disk configuration. The Fourier series are not convergent at $\rho = z = 0$; this is the point where the Ekman layer erupts into the $E^{\frac{1}{3}}$ layer, and feeds it. Moore & Saffman (1969) have shown how the matching conditions on the $E^{\frac{1}{4}}$ layer fields can be obtained by consideration of the admissible singularities of similarity solutions in the $E^{\frac{1}{3}}$ layer. They observe that the vertical velocity component can have a delta-function singularity at $(r, z) = (a, 0)$. The corresponding singularity in ψ is a step discontinuity, which we have taken into account by working separately in $r \gtrless a$ and then connecting.

In the stratified case, the singularity is in the buoyancy as well as in the hydrostatic layer. Pedlosky (1971) has shown in a similar context that the fluid entering the buoyancy layer at $z = 0$ is distributed in the $\sigma S^{\frac{1}{2}}$ layer over a very short distance, so that the buoyancy layer is absent except at $z = 0$ and 1.

The fact that a bottom slope discontinuity produces a shear layer of the Stewartson type has been demonstrated by Jacobs (1964) in his study of the Taylor column problem in a homogeneous fluid, whereas Davies (1972), in a beautiful series of experiments, studied the restriction of the column due to stratification. He finds that a very small amount of stratification reduces the height of the column significantly† (nonlinear effects were not negligible in his experiments). It follows directly from our theory that a strong Taylor column can still form if stratification is small enough, viz. if $\sigma S \ll E^{\frac{1}{2}}$. When $\sigma S \gg E^{\frac{1}{2}}$, the results (2.12)–(2.17) imply that there is no discontinuity in the interior and therefore no shear layers. This is, however, peculiar to our configuration: it is clear that a discontinuity in the *slope* of the bottom cannot produce any discontinuity in the interior owing to the diffusive nature of the flow when $S \gg E^{\frac{1}{2}}$. On the other hand, discontinuities in the *velocity* of the boundary will be felt in the interior. Such discontinuities will obviously be present in a typical experiment, where an obstacle is dragged along the bottom of a rotating tank. Therefore we conjecture that a Taylor column can still form in a stratified rotating fluid, when $E^{\frac{1}{2}} \ll \sigma S \ll 1$, its boundary surface consisting of layers of thickness $\sigma S^{\frac{1}{2}}$ and $E^{\frac{1}{2}}(\sigma S)^{-\frac{1}{4}}$. The detailed structure of those layers would of course require special investigation.

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† Unfortunately, Davies does not give the value of the Prandtl number σ in his experiments. Comparison with our theory is thus impossible.

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